Shared norms and the evolution of ethnic markers: Mathematical appendix

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1 Fitness derivations

Let W_{ijk} be the fitness of an individual with behavior i, marker j, in subpopulation k. Using the assumptions in the text, the expression is:

$$\begin{aligned} W_{ijk} &= e\Big(p_{ik}(1+\delta) + (1-p_{ik})(1)\Big) + (1-e)\left(\frac{x_{ijk}}{q_{jk}}(1+\delta) + \frac{q_{jk}-x_{ijk}}{q_{jk}}(1)\right) \\ &= e(1+\delta p_{ik}) + (1-e)\left(1+\delta\frac{x_{ijk}}{q_{jk}}\right). \end{aligned}$$

Collecting the δ terms,

$$W_{ijk} = 1+\delta\left(ep_{ik} + (1-e)\frac{x_{ijk}}{q_{jk}}\right).$$

Using the definition of the covariance, D, between i and j, we get:

$$\begin{aligned} W_{11k} &= 1+\delta\left(ep_{1k} + (1-e)\frac{p_{1k}q_{1k}+D_k}{q_{1k}}\right), \\ W_{00k} &= 1+\delta\left(ep_{0k} + (1-e)\frac{p_{0k}q_{0k}+D_k}{q_{0k}}\right), \\ W_{10k} &= 1+\delta\left(ep_{1k} + (1-e)\frac{p_{1k}q_{0k}-D_k}{q_{0k}}\right), \\ W_{01k} &= 1+\delta\left(ep_{0k} + (1-e)\frac{p_{0k}q_{1k}-D_k}{q_{1k}}\right). \end{aligned}$$

We use these to simplify expression (1) in the text. Although we do not mention it in the text, D is formally identical to the population genetics concept of linkage disequilibrium, which is just the covariance between alleles at two loci.

2 Derivation of Δp_{ik}

By definition $\Delta p_{1k} = \Delta x_{11k} + x_{10k}$. Assuming the replicator dynamic for transmission:

$$\Delta p_{1k} = x_{11k}(W_{11k} - \overline{W}_k) + x_{10k}(W_{10k} - \overline{W}_k).$$

This simplifies to:

$$\Delta p_{1k} = (1 - p_{ik})(x_{11k}W_{11k} + x_{10k}W_{10k}) - p_{ik}(x_{01k}W_{01k} + x_{00k}W_{00k}).$$

Substituting in fitness expressions in section 1 and using the definitions of x_{ijk} in terms of p, q, D, the above simplifies to:

$$\Delta p_{1k} = \delta \left(e p_{1k} p_{0k}(p_{1k} - p_{0k}) + \frac{1-e}{q_{1k}q_{0k}}(p_{1k}p_{0k}q_{1k}q_{0k} - D_k^2)(p_{1k} - p_{0k}) \right).$$

Factoring and simplifying further:

$$\Delta p_{1k} = \delta p_{1k} p_{0k}(p_{1k} - p_{0k}) \left(1 - (1-e)\frac{D_k^2}{p_{1k}p_{0k}q_{1k}q_{0k}} \right).$$

The above is the expression (2) in the text.

3 Derivation of Δq_{jk}

As with Δp_{ik}:

$$\begin{aligned} \Delta q_{1k} &= \Delta x_{11k} + \Delta x_{01k} \\ &= q_{0k}(x_{11k}W_{11k} + x_{01k}W_{01k}) - q_{1k}(x_{10k}W_{10k} + x_{00k}W_{00k}). \end{aligned}$$

Again, we make substitutions for fitness expressions and convert all x_{ijk} to be in terms of p, q, D. Simplifying:

$$\Delta q_{1k} = 2\delta D_k(p_{1k} - p_{0k}) \left(1 - \frac{e}{2} - (1-e)\frac{D_k}{q_{1k}q_{0k}} \left(\frac{q_{1k} - q_{0k}}{p_{1k} - p_{0k}} \right) \right).$$

The above is expression (3) in the text.

4 Derivation of ΔD_k

By definition, $\Delta D_k = D'_k - D_k$. Expanded, this is:

$$\Delta D_k = x_{11k}\frac{W_{11k}}{\overline{W}_k}x_{00k}\frac{W_{00k}}{\overline{W}_k} - x_{10k}\frac{W_{10k}}{\overline{W}_k}x_{01k}\frac{W_{01k}}{\overline{W}_k} - D_k.$$

Substituting fitness expressions and expressing all x_{ijk} in terms of p, q, D, then assuming all terms of order $D_k^3 \approx 0$, selection is weak such that $\overline{W}_k \approx 1$ and $\delta^2 \approx 0$ yields:

$$\Delta D_k = \delta\left(D_k(2p_{1k}p_{0k}(1-e)+1) + (1-e)\frac{D_k^2}{q_{1k}q_{0k}}(p_{1k}p_{0k})(q_{1k}q_{0k})\right).$$

Assuming D_k is small enough that $D_k^2 \approx 0$ gives expression (4) in the text.